Lecture 19:

Poisson Process; Exponential, Gamma, and Poisson Distributions

Part I:

Recall the definition of the Poisson Process

Let $N(t)$ represents the total number of occurance or events that have happened up to and including time t. We say that $\{N(s),\ s\geq 0\}$ is a (homogeneous) Poisson Process with rate λ if

(i). $N(0)=0$,

(ii). $N(t+s)-N(s) = \text{Poisson}(\lambda t)$, and

(iii). $N(t)$ has independent increments,

i.e., if $t_0<t_1<t_2<\cdots<t_n$,

then $N(t_1)-N(t_0),\ N(t_2)-N(t_1),\ \cdots,\ N(t_n)-N(t_{n-1})$ are mutually independent.

Remark 19.1. Time to first occurance in a Poisson Process is Exponential.

Since $N(s)$ represents the number of arrivals in $[0, s]$, by the definition of the Poisson Process, $N(s) = N(s) - N(0) = \text{Poisson}(\lambda s)$. This implies $\mathbb{P}(N(s)=0) = \mathbb{P}(\text{Poisson}(\lambda s)=0) = e^{-\lambda s}$.

Suppose the first arrival time is τ_1, then

$$\mathbb{P}(\tau_1 \leq s) = \mathbb{P}(N(s) \geq 1) = 1 - \mathbb{P}(N(s)=0) = 1 - e^{-\lambda s}.$$

Therefore, $\tau_1 = \text{exponential}(\lambda)$.

Remark 19.2. Similarly, after the first arrival had occurred, we can reset the counting process to count the event starting from τ_1. Then the time until next arrival is also an $\text{exponential}(\lambda)$. Moreover, the exponential interarrival times $\tau_1, \tau_2, \tau_3, \cdots$ are independent.

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Define $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$. Then, by Thm 17.2, $T_n \sim \text{gamma}(n, \lambda)$ is a gamma distribution.

Remark 17.3. Another way to calculate $\mathbb{P}(T_n \le t)$ (other that $\mathbb{P}(T_n \le t) = \int_0^t f_{T_n}(s)\,ds$) is

$$\mathbb{P}(T_n \le t) = \mathbb{P}(N(t) \ge n)$$
$$= \sum_{k=n}^{\infty} \mathbb{P}(N(t) = k)$$
$$= \sum_{k=n}^{\infty} e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

Thus "the survival"

$$\mathbb{P}(T_n > t) = 1 - \mathbb{P}(T_n \le t) = \sum_{k=0}^{n-1} e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}.$$

One can also check this formula by taking its derivative

$$\frac{d}{dt}\left(\mathbb{P}(T_n \le t)\right) = \sum_{k=n}^{\infty} \left(e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}\right)'$$
$$= \sum_{k=n}^{\infty} e^{-\lambda t} \cdot \lambda \cdot \left(\frac{(\lambda t)^{k-1}}{(k-1)!} - \frac{(\lambda t)^k}{k!}\right)$$
$$= e^{-\lambda t} \cdot \lambda \cdot \frac{(\lambda t)^{n-1}}{(n-1)!},$$

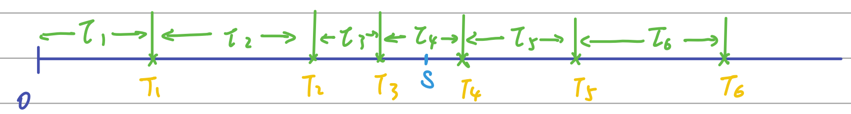
which coincides with the PDF of $\text{gamma}(n, \lambda)$.

Remark 19.4. In sum, given a Poisson Process $\{N(s): s \geq 0\}$ with rate $\lambda > 0$.

Counting Variable: $N(t) = N(t+s) - N(s) = \text{Poisson}(\lambda t)$

Interarrival Time: $\tau_i = \text{exponential}(\lambda)$, i.i.d.

Wait Time: $T_n = \sum_{i=1}^{n} \tau_i = \text{gamma}(n, \lambda)$.



$N(s) = \max\{n : T_n \leq s\}$

Part II. Constructing the Poisson Process

Let $\tau_1, \tau_2, \cdots$ be independent exponential(λ).

Let $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$ for $n \geq 1$; $T_0 = 0$.

Define $N(s) = \max\{n : T_n \leq s\}$

Here τ_n could be interpreted as time length between arrivals of students to Tim Hortons at DC. Thus, T_n would be the arrival time of the n-th student while $N(s)$ represents the number of arrivals by time s.

For example, $N(s) = n$ means $T_n \leq s < T_{n+1}$, i.e., the n-th student has arrived by time s, but the $(n+1)$-th has not.

Theorem 19.1. $\{N(s) : s \geq 0\}$ defined above is a Poisson Process with rate λ.

Lemma 19.1. $N(s) = \text{Poisson}(\lambda s)$.

Lemma 19.2. $N(t+s) - N(s) = \text{Poisson}(\lambda t)$ and is independent of $N(r)$, $0 \le r \le s$.

Lemma 19.3. $N(t)$ has independent increments.

Proof of Theorem 19.1.

①. $N(0)$ represents the number of arrivals at time 0. Since $T_0 = 0$ and

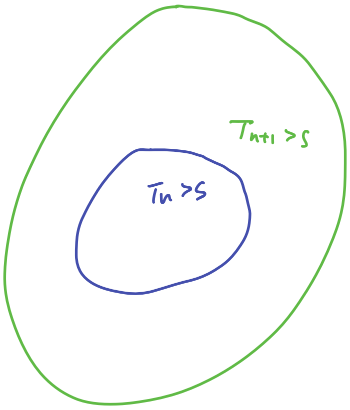
$$\mathbb{P}(T_1 = 0) = \mathbb{P}(\tau_1 = 0) = 1 - e^{-\lambda \cdot 0} = 0,$$

one has $\mathbb{P}(T_1 > 0 = T_0) = 1 - \mathbb{P}(T_1 = 0) = 1$.

That is, $\mathbb{P}(N(0) = 0) = 1$.

②. By Lemma 19.2 - 19.3, we see that $\{N(s) : s \ge 0\}$ is a Poisson Process with rate λ. $\square$

Proof of Lemma 19.1.



Notice that $\{N(s) = n\}$ means $\{T_n \le s < T_{n+1}\}$ and the set $\{T_{n+1} > s\} \supseteq \{T_n > s\}$. Thus,

$$
\begin{aligned}
&\{T_n \le s < T_{n+1}\} \\
&= \{T_n \le s,\ T_{n+1} > s\} \\
&= \{T_{n+1} > s\} \cap \{T_n \le s\} = \{T_{n+1} > s\} \setminus \left[\{T_{n+1} > s\} \cap \left(\{T_n \le s\}^c\right)\right] \\
&= \{T_{n+1} > s\} \setminus \left[\{T_{n+1} > s\} \cap \{T_n > s\}\right] \\
&= \{T_{n+1} > s\} \setminus \{T_n > s\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{P}(N(s) = n) &= \mathbb{P}(T_n \le s < T_{n+1}) \\
&= \mathbb{P}(\{T_{n+1} > s\} \setminus \{T_n > s\}) \\
&= \mathbb{P}(T_{n+1} > s) - \mathbb{P}(T_n > s) \\
&= \sum_{k=0}^{n} e^{-\lambda s} \cdot \frac{(\lambda s)^k}{k!} - \sum_{k=0}^{n-1} e^{-\lambda s} \cdot \frac{(\lambda s)^k}{k!} \\
&= e^{-\lambda s} \cdot \frac{(\lambda s)^n}{n!}
\end{aligned}
$$

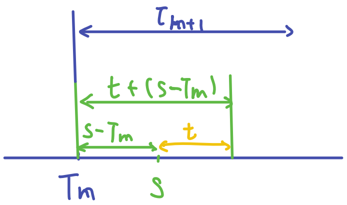
Since $T_n = \text{gamma}(n, \lambda)$

Thus, by definition, $N(s) = \text{Poisson}(\lambda s)$. $\square$

Proof of Lemma 19.2.

① Suppose $N(s) = m$, then $T_m \le s < T_{m+1}$.

For $\tau_{m+1} = T_{m+1} - T_m > s - T_m$ and $\tau_{m+1} = \text{exponential}(\lambda)$,



by the Lack of Memory Property, we have

$$\mathbb{P}(\tau_{m+1} > t + (s-T_m) \mid \tau_{m+1} > s - T_m) = \mathbb{P}(\tau_{m+1} > t).$$

Thus, $\mathbb{P}(\tau_{m+1} \le t + (s-T_m) \mid \tau_{m+1} > s - T_m) = \mathbb{P}(\tau_{m+1} \le t)$,

i.e., $\mathbb{P}(s - T_m < \tau_{m+1} \le t + (s - T_m)) = \mathbb{P}(\tau_{m+1} \le t)$.

Therefore, $\tau_{m+1} - (s - T_m) = \text{exponential}(\lambda)$. Denote it by $\tilde{\tau}_1$.

Denote $\tilde{\tau}_i = \tau_{m+i}$, for $i = 2, 3, \cdots$; $S_n = \tilde{\tau}_1 + \tilde{\tau}_2 + \cdots + \tilde{\tau}_n$;

$S_0 = 0$. Then, $\tilde{\tau}_i = \text{exponential}(\lambda)$ and $S_n = \text{gamma}(n, \lambda)$.

Thus, $\mathbb{P}(N(t+s) - N(s) = n)$

$$= \mathbb{P}(N(t+s) = n+m)$$

$$= \mathbb{P}(T_{n+m} \le t+s < T_{n+m+1})$$

$$= \mathbb{P}(T_m + \tau_{m+1} + \cdots + \tau_{m+n} \le t+s < T_m + \tau_{m+1} + \cdots + \tau_{m+n+1})$$

$$= \mathbb{P}([\tau_{m+1} - (s - T_m)] + \tau_{m+2} + \cdots + \tau_{m+n} \leq t < [\tau_{m+1} - (s - T_m)] + \tau_{m+2} + \cdots + \tau_{m+n+1})$$

$$= \mathbb{P}(\tilde{\tau}_1 + \tilde{\tau}_2 + \cdots + \tilde{\tau}_n \leq t < \tilde{\tau}_1 + \tilde{\tau}_2 + \cdots + \tilde{\tau}_{n+1})$$

$$= \mathbb{P}(S_n \leq t < S_{n+1})$$

Same as in the proof of Lemma 19.1

$$= \mathbb{P}(S_{n+1} > t) - \mathbb{P}(S_n > t)$$

$$= \sum_{k=0}^{n} e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!} - \sum_{k=0}^{n-1} e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \mathbb{P}(\text{Poisson}(\lambda t) = n)$$

Thus, $N(t+s) - N(s) = \text{Poisson}(\lambda t)$.

②. $\forall\ r \in [0, s]$, $N(r) \leq m$.

For $k < m$, one has $\tau_1, \cdots, \tau_{k+1}, \tilde{\tau}_1 = \tau_{m+1} - (s - T_m), \tilde{\tau}_2, \cdots, \tilde{\tau}_{n+1}$ are independent. Thus,

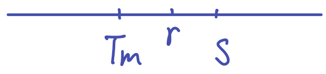
$$\mathbb{P}(N(t+s) - N(s) = n,\ N(r) = k)$$

$$= \mathbb{P}(T_{m+n} \leq t + s < T_{m+n+1},\ T_k \leq r < T_{k+1})$$

$$= \mathbb{P}(\tilde{\tau}_1 + \cdots + \tilde{\tau}_n \leq t + s < \tilde{\tau}_1 + \cdots + \tilde{\tau}_{n+1},\ \tau_1 + \cdots + \tau_k \leq r < \tau_1 + \cdots + \tau_{k+1})$$

$= \mathbb{P}(\tilde{\tau}_1 + \cdots + \tilde{\tau}_n \le t+s < \tilde{\tau}_1 + \cdots + \tilde{\tau}_{n+1}) \cdot \mathbb{P}(\tau_1 + \cdots + \tau_k \le r < \tau_1 + \cdots + \tau_{k+1})$

$= \mathbb{P}(N(t+s) - N(s) = n) \cdot \mathbb{P}(N(r) = k)$



For $k = m$, since $N(s) = m$ and $r \le s$, we have

$\{N(r) = m\} = \{T_m \le r\}$ and

$\mathbb{P}(N(t+s) - N(s) = n, \ N(r) = m)$

$= \mathbb{P}(T_{m+n} \le t+s < T_{m+n+1}, \ T_m \le r)$

$= \mathbb{P}(\tilde{\tau}_1 + \cdots + \tilde{\tau}_n \le t+s < \tilde{\tau}_1 + \cdots + \tilde{\tau}_{n+1}, \ \tau_1 + \cdots + \tau_m \le r)$

$= \mathbb{P}(\tilde{\tau}_1 + \cdots + \tilde{\tau}_n \le t+s < \tilde{\tau}_1 + \cdots + \tilde{\tau}_{n+1}) \cdot \mathbb{P}(\tau_1 + \cdots + \tau_m \le r)$

$= \mathbb{P}(N(t+s) - N(s) = n) \cdot \mathbb{P}(N(r) = m)$.

Thus, $\forall \ r \in [0, s]$, $0 \le k \le m$,

$\mathbb{P}(N(t+s) - N(s) = n, \ N(r) = k)$

$= \mathbb{P}(N(t+s) - N(s) = n) \cdot \mathbb{P}(N(r) = k)$.

This implies, $N(t+s) - N(s)$ and $N(r)$ are independent for any $t \ge 0$ and $0 \le r \le s$.

Proof of Lemma 19.3 (Proof by mathematical induction).

Statement. $\forall\ 0=t_0<t_1<\cdots<t_n$, $N(t_1)-N(t_0)$, $N(t_2)-N(t_1), \cdots, N(t_n)-N(t_{n-1})$ are mutually independent.

Lemma 19.2 implies $N(t_2)-N(t_1)$ and $N(t_1)=N(t_1)-N(t_0)$ are independent. So the statement is true for $n=2$.

Suppose the statement holds for $n\geq 2$. Then for the case $n+1$, we want to show

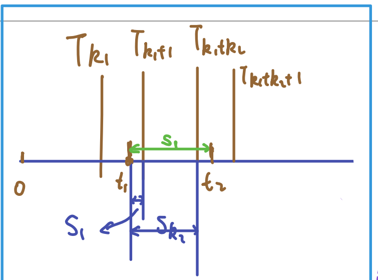
$$\mathbb{P}(N(t_1)-N(t_0)=k_1, N(t_2)-N(t_1)=k_2, \cdots, N(t_{n+1})-N(t_n)=k_{n+1})$$

$$=\mathbb{P}(N(t_1)-N(t_0)=k_1)\cdot\mathbb{P}(N(t_2)-N(t_1)=k_2)\cdot\cdots\cdot\mathbb{P}(N(t_{n+1})-N(t_n)=k_{n+1}),$$

$\forall k_1, k_2, \cdots, k_{n+1}\in\mathbb{N}$.

To see this, let $s_1=t_2-t_1$, $s_2=t_3-t_1$, $\cdots$, $s_n=t_{n+1}-t_1$, and $S_i=T_{k_1+i}-t_1$, $\forall i\in\mathbb{N}$.

Thus,



$$S_i = T_{k_1+i} - t_1$$

$$\mathbb{P}(N(t_1)-N(t_0)=k_1,\ N(t_2)-N(t_1)=k_2,\ \cdots,\ N(t_{n+1})-N(t_n)=k_{n+1})$$

$$= \mathbb{P}(N(t_1)=k_1,\ N(t_2)-N(t_1)=k_2,\ \cdots,\ N(t_{n+1})-N(t_n)=k_{n+1})$$

By Lemma 19.2

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(N(t_2)-N(t_1)=k_2,\ \cdots,\ N(t_{n+1})-N(t_n)=k_{n+1})$$

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(N(t_2)=k_1+k_2,\ N(t_3)=k_1+k_2+k_3,\ \cdots,\ N(t_{n+1})=k_1+k_2+\cdots+k_{n+1})$$

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(T_{k_1+k_2}\le t_2 < T_{k_1+k_2+1},\ \cdots,\ T_{k_1+k_2+\cdots+k_{n+1}} \le t_{n+1} \le T_{k_1+k_2+\cdots+k_{n+1}+1})$$

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(S_{k_2} \le \underbrace{t_2-t_1}_{s_1} < S_{k_2+1},\ \cdots,\ S_{k_2+\cdots+k_{n+1}} \le \underbrace{t_{n+1}-t_1}_{s_n} \le S_{k_2+\cdots+k_{n+1}+1})$$

By the Lack of Memory Property

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(N(s_1)=k_2,\ \cdots,\ N(s_n)-N(s_{n-1})=k_{n+1})$$

By induction

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(N(s_1)=k_2)\cdots \mathbb{P}(N(s_n)-N(s_{n-1})=k_{n+1})$$

$$= \mathbb{P}(N(t_1)=k_1)\cdot \mathbb{P}(N(t_2)-N(t_1)=k_2)\cdot \cdots \cdot \mathbb{P}((N(t_{n+1})-N(t_1))-(N(t_n)-N(t_1))=k_{n+1}).$$

$$= \mathbb{P}(N(t_1)-N(t_0)=k_1)\cdot \mathbb{P}(N(t_2)-N(t_1)=k_2)\cdots \mathbb{P}(N(t_{n+1})-N(t_n)=k_{n+1}). \qquad \square$$

Definition 19.1 (Renewal Process: a generalization of Poisson Process)

$\{R(t): t\ge 0\}$ is a Renewal Process if $\exists$ positive i.i.d. random variables $(X_\ell)_{\ell\ge1}$ with $0<\mathbb{E}X_\ell<\infty$, such that

$$R(t) = \max\{n : \sum_{\ell=1}^{n} X_\ell \le t\}.$$

This is the end of this lecture!

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